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LUMPED PARAMETER REPRESENTATION OF A LONGITUDINALLY VIBRATING ELASTIC ROD VISCOUSLY DAMPED IN-SPAN

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1. Introduction

Although it is perhaps not immediately obvious, there is a very intimate relation between discrete and continuous systems. In fact, they generally represent two distinct mathematical models of the same physical system [1]. To demonstrate this, the differential equation for the transverse vibrations of a string is derived first by regarding it as a discrete system and then letting it approach a continuous model in the limit. Then the problem is formulated by regarding the system as continuous from the beginning, and it is shown that the same equation of motion is obtained in both cases. In reference [2], after deriving the equations of longitudinal oscillations of mass points connected by massless springs and transverse planar oscillations of mass points on a stretched massless string, it is observed that both equations have exactly the same form. Then, the behaviour of these discrete systems is examined when the characteristic scale of the phenomena is large compared with the interparticle spacing. Noting that the resulting limiting equations describe a continuous string, the same equations are also derived directly. A similar line of thought is followed also in the following two works. In reference [3], by starting from the loaded string discrete system, the differential equation for the vibrations of the wire is derived by making the number of discrete masses tend to infinity in the discrete equation of motion. On the other hand, in reference [4], the transition approach from a discrete to a continuous system is applied to the small longitudinal vibrations of an infinitely long elastic rod. In reference [5], in the section entitled "Lumped Parameter Representation of Continuous Systems", a slightly different path is followed to investigate the accuracy with which the frequencies of longitudinal vibrations of a continuous rod may be estimated by representing it as a series of identical masses and springs where the rod under consideration is assumed to be rigidly held at one end and fixed at the other. After establishing an explicit formula for the eigenfrequencies of the discrete system, it is shown, among others, that the discrete mass approximation underestimates the natural frequencies of the continuous system.

The present letter deals with the mechanical system shown in Figure 1, which is essentially the same as that in reference [5], i.e. a longitudinally vibrating rod fixed at one end and free at the other. In the present study a viscous damping element is included in the system, a feature which is not considered in the references cited above. The first step in this work aims to derive the characteristic equation of this continuous system (to the knowledge of the authors, it is not available in the technical literature). The second step, which results from an aim that is considered to be more important by the authors, is to obtain the characteristic values and to study the dependence of their convergence properties towards the actual values, with the number of discrete masses n, for a continuous system approximated by a uniform chain comprised of n equal masses and springs. It is thought that a contribution will result in the area of investigation of the approximation properties of a discrete model for a damped continuous system. These kinds

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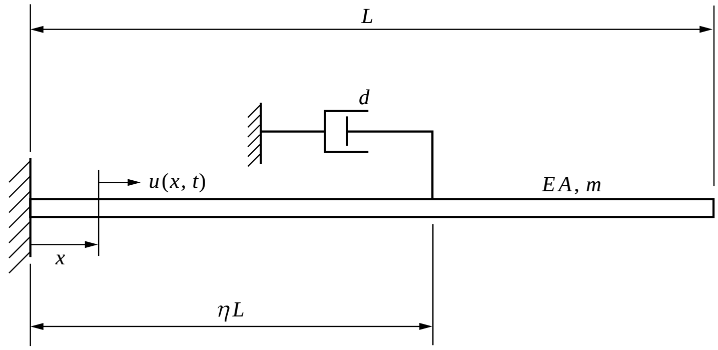

Figure 1. Original system: longitudinally vibrating elastic rod, viscously damped in-span.

of damped continuous systems could be encountered in diverse technological areas such as drill strings, ocean cables, piping systems, and space structures.

2. THEORY

The mechanical system to be investigated is shown in Figure 1. It consists of an axially vibrating fixed-free elastic rod of length L which is damped viscously at an intermediate location ηL. Axial rigidity and mass per unit length of the rod are EA and m, respectively. The effective damping constant is d. Consider first the continuous model.

2.1. *The continuous model*

The equations of motion of the longitudinal vibrations of a rod is the well known partial differential equation [1].

$$EAu''(x, t) = m\ddot{u}(x, t) \tag{1}$$

where over dots and primes denote partial derivatives with respect to time t and position coordinate x, respectively. Denote the axial displacements in the regions to the left and right of the attachment point of the damper as $u_1(x, t)$ and $u_2(x, t)$. Both of them are subject to the differential equation (1). The corresponding boundary and matching conditions are:

$$u_1(0, t) = 0, \qquad u_1(\eta L, t) = u_2(\eta L, t), \qquad u_2'(L, t) = 0$$

$$u_1'(\eta L, t) - u_2'(\eta L, t) + [d/(EA)]\dot{u}_1(\eta L, t) = 0. \tag{2}$$

Assuming a solution of the type

$$u_i(x, t) = U_i(x) \exp(\lambda t) \qquad (i = 1, 2) \tag{3}$$

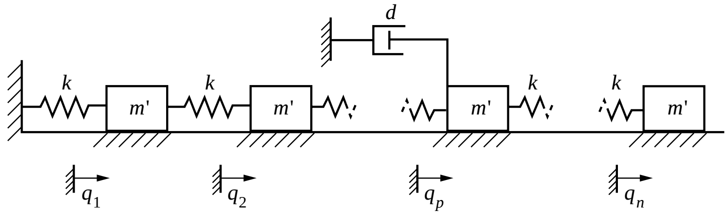

Figure 2. Discrete model: viscously damped uniform oscillator made up of n equal masses and springs.

TABLE 1

Dependence of the first four dimensionless eigenfrequencies of the undamped rod, on the number of discrete masses n

	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$
n	1·57079633	4·71238898	7·85398163	10·99557429
1	1·00000000	–	–	–
2	1·23606798	3·23606798	–	–
3	1·33512560	3·74093881	5·40581321	–
4	1·38918542	4·00000000	6·12835554	7·51754097
5	1·42314838	4·15415013	6·54860734	8·41253533
6	1·44644016	4·25525864	6·81677696	8·98212898
7	1·46339849	4·32623792	7·00000000	9·36782849
8	1·47629375	4·37860784	7·13181369	9·64215418
9	1·48642822	4·41873877	7·23051764	9·84506685
10	1·49460187	4·45041868	7·30682049	10·00000000
15	1·51947507	4·54283333	7·51957597	10·41915759
20	1·53210935	4·58733702	7·61564437	10·59926009
30	1·54487482	4·63052773	7·70390131	10·75684553
40	1·55130654	4·65158631	7·74486966	10·82650398
50	1·55518119	4·66403904	7·76838473	10·86521501
100	1·56296551	4·68851472	7·81291859	10·93541388
150	1·56557063	4·69654134	7·82700044	10·95660692
200	1·56687512	4·70052919	7·83389476	10·96677950
250	1·56765844	4·70291367	7·83798398	10·97274609
300	1·56818090	4·70449986	7·84069027	10·97666644
350	1·56855422	4·70563116	7·84261358	10·97943850
400	1·56883428	4·70647870	7·84405072	10·98150208
500	1·56922646	4·70766391	7·84605499	10·98436880
600	1·56948797	4·70845318	7·84738617	10·98626546
800	1·56981494	4·70943878	7·84904448	10·98861996
1000	1·57001116	4·71002961	7·85003645	10·99002394
1200	1·57014199	4·71042328	7·85069650	10·99095629
1500	1·57027283	4·71081677	7·85135555	10·99188572

$U_i(x)$ and λ being the unknown amplitude functions and characteristic value, results in the following ordinary differential equations for $U_i(x)$

$$U_i''(x) - \beta^2 U_i(x) = 0 \qquad (i = 1, 2) \tag{4}$$

where

$$\beta^2 = m\lambda^2/(EA) \tag{5}$$

is introduced.

The solutions of the differential equations (4) are

$$U_1(x) = C_1 \exp(\beta x) + C_2 \exp(-\beta x)$$
$$U_2(x) = C_3 \exp(\beta x) + C_4 \exp(-\beta x) \tag{6}$$

where C_1–C_4 are integration constants to be determined. The substitution of the solutions (3) in connection with (6) into the boundary conditions (2) yields a set of four homogeneous equations for the determination of the constants C_1–C_4. For non-vanishing solutions, the determinant of the coefficients must be equated to zero. It can be shown

TABLE 2

Dependence of the first four dimensionless eigencharacteristics on the number of discrete masses n

n	$\overline{\beta}_1$	$\overline{\beta}_2$	$\overline{\beta}_3$	$\overline{\beta}_4$
	−0·00147243 + 1·57079601i	−0·00021482 + 4·71238901i	−0·00224967 + 7·85398163i	−0·00021482 + 10·99557426i
1	−0·00112483 + 0·99999937i	—	—	—
2	−0·00062179 + 1·23606810i	−0·00162787 + 3·23606684i	—	—
3	−0·00117869 + 1·33512539i	−0·00036301 + 3·74093886i	−0·00183280 + 5·40581157i	—
4	−0·00082623 + 1·38918552i	−0·00149978 + 3·99999972i	−0·00023392 + 6·12835557i	−0·00193940 + 7·51753884i
5	−0·00116810 + 1·42314825i	−0·00059778 + 4·15415018i	−0·00169222 + 6·54860700i	−0·00016233 + 8·41253534i
6	−0·00140650 + 1·44643981i	−0·00011893 + 4·25525866i	−0·00204645 + 6·81677697i	−0·00044848 + 8·98212882i
7	−0·00115958 + 1·46339839i	−0·00072542 + 4·32623797i	−0·00157477 + 6·99999982i	−0·00034736 + 9·36782851i
8	−0·00134838 + 1·47629349i	−0·00027630 + 4·37860788i	−0·00209931 + 7·13181353i	−0·00007149 + 9·64215418i
9	−0·00115363 + 1·48642815i	−0·00080403 + 4·41873881i	−0·00149369 + 7·23051753i	−0·00048279 + 9·84506687i
10	−0·00130965 + 1·49460166i	−0·00040334 + 4·45041873i	−0·00203645 + 7·30682029i	0·00000000 + 10·00000000i
15	−0·00136140 + 1·51947482i	−0·00033857 + 4·54283337i	−0·00212718 + 7·51957583i	−0·00002228 + 10·41915758i
20	−0·00138819 + 1·53210909i	−0·00030670 + 4·58733706i	−0·00216593 + 7·61564426i	−0·00005114 + 10·59926008i
30	−0·00141561 + 1·54487454i	−0·00027535 + 4·63052776i	−0·00219961 + 7·70390123i	−0·00009259 + 10·75684552i
40	−0·00142957 + 1·55130626i	−0·00025991 + 4·65158635i	−0·00221438 + 7·74486960i	−0·00011817 + 10·82650397i
50	−0·00143802 + 1·55518090i	−0·00025074 + 4·66403907i	−0·00222255 + 7·76838468i	−0·00013510 + 10·86521500i
100	−0·00145510 + 1·56296521i	−0·00023262 + 4·68851475i	−0·00223724 + 7·81291857i	−0·00017255 + 10·93541386i
150	−0·00146085 + 1·56557032i	−0·00022665 + 4·69654137i	−0·00224164 + 7·82700043i	−0·00018610 + 10·95660689i
200	−0·00146373 + 1·56687481i	−0·00022368 + 4·70052922i	−0·00224375 + 7·83389475i	−0·00019308 + 10·96677948i
250	−0·00146547 + 1·56765813i	−0·00022190 + 4·70291370i	−0·00224498 + 7·83798397i	−0·00019733 + 10·97274606i
300	−0·00146663 + 1·56818059i	−0·000221072 + 4·70449989i	−0·00224579 + 7·84069026i	−0·00020020 + 10·97666641i
350	−0·00146745 + 1·56855391i	−0·00021987 + 4·70563119i	−0·00224636 + 7·84261358i	−0·00020225 + 10·97943847i
400	−0·00146807 + 1·56883397i	−0·00021924 + 4·70647873i	−0·00224678 + 7·84405072i	−0·00020380 + 10·98150205i
500	−0·00146894 + 1·56922614i	−0·00021836 + 4·70766394i	−0·00224737 + 7·84605499i	−0·00020598 + 10·98436877i
600	−0·00146952 + 1·56948766i	−0·00021777 + 4·70845321i	−0·00224776 + 7·84738616i	−0·00020744 + 10·98626543i
800	−0·00147025 + 1·56981463i	−0·00021703 + 4·70943881i	−0·00224824 + 7·84904448i	−0·00020928 + 10·98861993i
1000	−0·00147068 + 1·57001085i	−0·00021659 + 4·71002964i	−0·00224853 + 7·85003645i	−0·00021038 + 10·99002391i
1200	−0·00147097 + 1·57014167i	−0·00021629 + 4·71042331i	−0·00224872 + 7·85069650i	−0·00021111 + 10·99095626i
1500	−0·00147126 + 1·57027252i	−0·00021600 + 4·71081680i	−0·00224891 + 7·85135555i	−0·00021186 + 10·99188570i

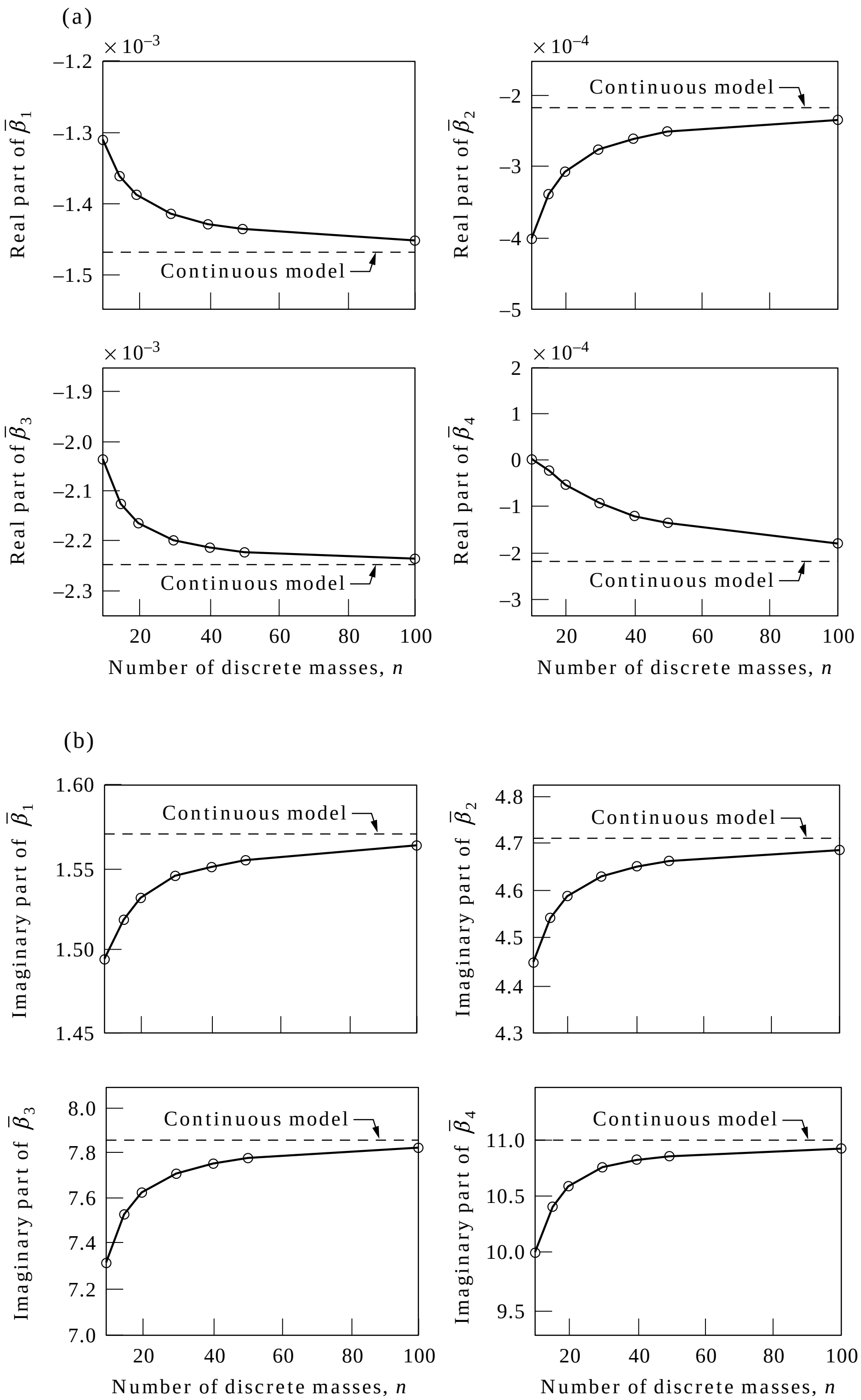

Figure 3. Dependence of (a) the real parts, (b) the imaginary parts of the eigencharacteristics $\bar{\beta}_i$ on the number of the discrete masses.

after some algebraic manipulations that this condition results in the following characteristic equation

$$2\cosh\overline{\beta} + a\{\sinh\overline{\beta} - \sinh[(1-2\eta)\overline{\beta}]\} = 0 \tag{7}$$

where the abbreviations

$$\overline{\beta} = \beta L, \qquad a = d\lambda/(EA\beta) \tag{8}$$

are introduced. For comparison purposes with other studies, it is worth noting that $\overline{\beta}$ above can also be written as

$$\overline{\beta} = \lambda L/c \tag{9}$$

where $c = \sqrt{E/\rho}$ represents the velocity of the wave propagation along the rod, ρ being the density of the rod material.

The complex roots of the transcendental equation (7) give the dimensionless characteristic parameters $\overline{\beta}$ and therefore by considering expressions (8), the eigencharacteristics λ of the system in Figure 1.

Before proceeding further to the solution of the complex equation (7), it is in order to consider the special case $\eta = 1$ which corresponds to the case that the rod is damped at the free end. It is an easy matter to show that the equation (7) simplifies to the expression

$$(EA/c)\exp(\overline{\beta}) + \exp(-\overline{\beta}) = -d(\exp(\overline{\beta}) - \exp(-\overline{\beta})) \tag{10}$$

which is also given in [6] in different notations.

With $\overline{\beta}$ as a complex number

$$\overline{\beta} = x + iy \tag{11}$$

it can be shown after manipulations that the solution of the complex equation (7) with respect to $\overline{\beta}$ is equivalent to the simultaneous solution of the following set of two real equations with respect to x and y

$$(2\cosh x + a\sinh x)\cos y - a\sinh[(1-2\eta)x]\cos[(1-2\eta)y] = 0$$
$$(2\sinh x + a\cosh x)\sin y - a\cosh[(1-2\eta)x]\sin[(1-2\eta)y] = 0. \tag{12}$$

2.2. *The discrete model*

The original continuous system in Figure 1 is now modelled as a uniform oscillator consisting of n equal springs of stiffness coefficient k and n equal masses of mass m' (Figure 2). The oscillator is damped at the pth mass by a viscous damper of damping coefficient d.

It is reasonable to take

$$k = n(EA/L), \qquad m' = mL/n \tag{13}$$

where p denotes the integer nearest to ηn.

TABLE 3

Minimum number of discrete masses required for the given error bounds

Error	real ($\overline{\beta}_1$)	imag ($\overline{\beta}_1$)	real ($\overline{\beta}_2$)	imag ($\overline{\beta}_2$)	real ($\overline{\beta}_3$)	imag ($\overline{\beta}_3$)	real ($\overline{\beta}_4$)	imag ($\overline{\beta}_4$)
5%	30	10	200	15	20	15	500	20
2%	100	30	500	30	40	30	1200	40
1%	150	50	1000	100	100	100	–	100
0·5%	250	100	–	150	150	150	–	150
0·2%	600	250	–	300	300	300	–	300
0·1%	1200	500	–	600	600	600	–	600

The mass, damping and stiffness matrices **M**, **C**, and **K** of the system in Figure 2 are simply

$$\mathbf{M} = \mathrm{diag}\,(m'), \qquad \mathbf{C} = \begin{bmatrix} & \vdots^{\,p} & \\ & \vdots & \mathbf{0} \\ p \;\cdots\cdots & d & \\ \mathbf{0} & & \end{bmatrix},$$

$$\mathbf{K} = k \begin{bmatrix} 2 & -1 & & & \mathbf{0} \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ \mathbf{0} & & & -1 & 1 \end{bmatrix}. \tag{14}$$

Going to the state description of the system, it can be shown that the eigencharacteristics of the damped oscillator can be obtained as the eigenvalues λ of the following $2n \times 2n$ system matrix

$$\mathbf{A} = \left[\begin{array}{c:c} \mathbf{0} & \mathbf{I} \\ \hdashline -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{array}\right] \tag{15}$$

which can be written as

$$\mathbf{A} = \left[\begin{array}{ccccc:ccc} & & & & & 1 & & \mathbf{0} \\ & & \mathbf{0} & & & & \ddots & \\ & & & & & \mathbf{0} & & \ddots \\ & & & & & & p & 1 \\ \hdashline -2\omega_0^2 & \omega_0^2 & & & & \ddots & \vdots & \\ \omega_0^2 & -2\omega_0^2 & \omega_0^2 & \mathbf{0} & & & \ddots\vdots & \mathbf{0} \\ & \ddots & \ddots & \ddots & p\cdots & \cdots\cdots & \cdots & -v \\ & & -2\omega_0^2 & \omega_0^2 & & & \mathbf{0} & \\ \mathbf{0} & & \ddots & \ddots & & & & \\ & & \omega_0^2 & -\omega_0^2 & & & & \end{array}\right] \tag{16}$$

where

$$\omega_0^2 = k/m' = n^2EA/(mL^2), \qquad v = d/m' = nd/(mL) \tag{17}$$

and **I** being the n dimensional identity matrix.

Once the eigenvalues λ of the above system matrix $\mathbf{A}$ are determined, the resulting complex numbers multiplied by L/c can be compared with those $\overline{\beta}$ obtained as the roots of equation (7) or equivalently, as the solutions of the set of equations (12) in connection with (11).

For the sake of completeness, the corresponding expressions for the undamped case are also collected. As is known, in the undamped case even the explicit expressions of the eigenfrequencies can be given for both systems, rather than the frequency equations. The ith eigenfrequency of the system in Figure 1 without damping is [1]

$$\omega_i = (i - 1/2)\pi\sqrt{EA/(mL^2)} \tag{18}$$

whereas that of the system in Figure 2 can be shown to be [7]

$$\omega_i = \sqrt{2n^2\{1 - \cos[(2i-1)\pi/(2n+1)]\}}\sqrt{EA/(mL^2)}. \tag{19}$$

It is an easy matter to show that when n tends to infinity, formula (19) reduces to the expression given by (18).

3. NUMERICAL APPLICATIONS

This section is devoted to the numerical evaluation of the expressions used and derived in the preceding section. The following data are selected for the physical parameters of the vibrating system in Figure 1. The rod under consideration is an aluminum rod of circular section.

$$L = 1\ \text{m}, \qquad A = \pi \times 10^{-4}\, m^2, \qquad E = 7 \times 10^{10}\ \text{N/m}^2,$$

$$\rho = 2{\cdot}86 \times 10^3\ \text{kg/m}^3, \qquad d = 10\ \text{N/(m/s)}, \qquad \eta = 0{\cdot}6.$$

Table 1 is concerned with the undamped case. The first row contains the first four dimensionless exact eigenfrequencies $\bar{\omega}_i = \omega_i/\sqrt{EA/(mL^2)}$ of the continuous rod, calculated from equation (18). In the remainder of the table, the corresponding non-dimensional eigenfrequencies of the discretized rod, i.e. uniform oscillator are collected in dependence of the number of the selected degrees of freedom n, calculated from (19). In Table 1, it can be seen that the discrete approximation error decreases with number of discrete masses n, with initially a larger sensitivity with respect to n, and approaching zero asymptotically. It is worth noting that with some of the smaller n values, there is the additional error producing effect due to the rounding of the number ηn as it may not coincide with a mass point, i.e. it may not be an integer, in which case it is rounded to the nearest integer. As expected, convergence is best for the fundamental frequency. The sensitivity of the error with respect to n becomes generally less for the higher frequencies.

Consider now the damped case. The corresponding values which are complex numbers rather than real are collected in Table 2. The non-dimensional eigencharacteristics $\overline{\beta}$ are given rather than λ, to allow comparison with the undamped case. The first row contains $\overline{\beta}$ values which were obtained from the numerical solution of the set of equations given in (12) with the help of MATLAB. In other words, these are the "exact" values. The complex numbers in the remainder of the table show the corresponding $\overline{\beta}$ values which were determined as the eigenvalues of the system matrix $\mathbf{A}$ in (16) multiplied by L/c according to the relation (9). In other words, these represent the approximate values in dependence of the degrees of freedom of the uniform oscillator. It is worth noting that in case of n degrees of freedom, the eigenvalues are computed from a matrix of order $2n \times 2n$. Due to the smallness of the selected viscous damping constant d, the imaginary parts of the $\overline{\beta}_i$ values are practically the same as the $\overline{\omega}_i$ values in Table 1, providing an indirect indication of the validity of the method used to calculate the damped eigencharacteristics. The convergence characteristics of the damped case are observed to be similar to those of the undamped case, including the rounding effect associated with ηn, and the dependence of the sensitivity of the error with respect to n, on the mode number. In Figure 3, the real and imaginary parts of the eigencharacteristics $\bar{\beta}$ for the damped case are plotted for a certain range of the number of discrete masses n, to illustrate the nature of convergence with n. In Figure 3(a), it can be seen that the discrete mass approximation overestimates the real part of the eigencharacteristics except for the second mode. On the other hand, in Figure 3(b), it can be observed that the discrete mass approximation underestimates the imaginary parts, i.e. the "damped eigenfrequencies" for all four modes. It is worth noting that what is shown in reference [5] for the undamped case, remains valid also for the damped case. To see the dependence of the error on the number of discrete masses better, minimum numbers of discrete masses required for some prescribed error bounds are given in Table 3.

Table 3 is based on the same set of discrete mass numbers n, which were used in Table 2. The columns for the imaginary parts in this table will be the same as the columns of a table for the undamped case because of the smallness of the damping coefficient, as can be seen from the comparison of Tables 1 and 2. It can be observed from Table 3 that the errors in the real parts of $\bar{\beta}_1$ and $\bar{\beta}_3$ are reduced much faster with increasing n, compared to those of $\bar{\beta}_2$ and $\bar{\beta}_4$. On the other hand, the decrease of errors in all the imaginary parts are more sensitive to an increase in n, than those in the real parts. Furthermore, all the imaginary part errors vary with n in roughly the same manner.

4. CONCLUSIONS

The present study deals with a longitudinally vibrating elastic rod fixed at one end free at the other, damped viscously by a single damper in-span. In the first step, the characteristic equation of the continuous system is derived. In the second, the rod is modelled as a uniform oscillator consisting of n equal masses and springs. The main purpose of the work was to study the dependence of the convergence of the uniform oscillator model eigencharacteristics towards those obtained from the continuous system (i.e. the "exact" values), on the number of discrete masses n.

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